



On symmetric nonlocal games[☆]

Andris Ambainis, Dmitry Kravchenko^{*}, Nikolay Nahimov, Alexander Rivosh, Madars Virza¹

Faculty of Computing, University of Latvia, Latvia

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ABSTRACT

Nonlocal games are used to display differences between the classical and quantum world. In this paper, we study symmetric XOR games, which form an important subset of nonlocal games. We give simple methods for calculating the classical and the quantum values for symmetric XOR games with one-bit input per player. We illustrate those methods with two examples. One example is an N -player game (due to Ardehali (1992) [3]) that provides the maximum quantum-over-classical advantage. The second example comes from generalization of CHSH game by letting the referee to choose arbitrary symmetric distribution of players' inputs.

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1. Introduction

1.1. Nonlocal games

A *nonlocal game* is a cooperative game of two or more players. Nonlocal games are commonly studied in the context of quantum mechanics, to illustrate the differences between the classical and quantum world.

We can view nonlocal games as games between a *referee* and N *players*, where all communication is between the referee and players. The referee chooses the settings of the game by giving some information (or *input*) x_i to each of the players. Each player must return an answer (or *output*) y_i . Players win if the answers $y_1, y_2, \dots$ satisfy a certain condition $f(x_1, x_2, \dots, y_1, y_2, \dots)$ set at the beginning of the game.

Players are not allowed to communicate among themselves. In the classical case, they can use shared common bits that are independent of the inputs $x_1, \dots, x_N$. In the quantum case, they can use a common N -partite quantum state $|\psi\rangle$ that is independent of $x_1, \dots, x_N$. Each player receives a part of $|\psi\rangle$ and is allowed to apply a measurement that depends on his input x_i to this part of $|\psi\rangle$. The player's answer y_i is determined by the outcome of this measurement, together with x_i .

A very famous example of a nonlocal game is the CHSH game [6,7]. In this game, there are two players who receive binary inputs $x_1, x_2 \in \{0, 1\}$. They must produce outputs $y_1, y_2 \in \{0, 1\}$. If at least one of x_i 's is equal to 0, players win if they answer $y_1, y_2 : y_1 = y_2$. If $x_1 = x_2 = 1$, players win if they answer $y_1, y_2 : y_1 \neq y_2$.

In the CHSH game, a quantum strategy that uses a shared quantum state $|\psi\rangle$ can win with probability $\approx 85.35\%$ while any classical strategy has at most a 75% probability of winning. Quantum strategies get their advantage through the non-classical correlations of the measurement outcomes of different parts of quantum system $|\psi\rangle$ (these correlations are called *entanglement* or *nonlocality*).

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^{*} Corresponding author.

E-mail address: kravchenko@gmail.com (D. Kravchenko).

¹ Since September 2011, at the Department of EECS, Massachusetts Institute of Technology.

In general, the maximum winning probability in a nonlocal game is hard to compute. It is NP-hard to compute it for 2-player games with quantum inputs and outputs or 3-player games classically [10].

1.2. Nonlocal XOR games

XOR games are the most widely studied class of nonlocal games. In a XOR game, players' outputs $y_1, y_2, \dots, y_N$ are 0–1 valued. The winning condition can depend only on $x_1, x_2, \dots, x_N$ and $y_1 \oplus y_2 \oplus \dots \oplus y_N$. The simplest particular case is $N = 2$. Then, a dependence on $y_1 \oplus y_2$ is equivalent to dependence on whether $y_1 = y_2$ or $y_1 \neq y_2$. XOR games include the several well studied nonlocal games (for example, the CHSH game).

Let p_q be the winning probability of the best quantum strategy and p_{cl} be the winning probability of the best classical strategy. In the context of nonlocal games, one typically studies the difference between the best winning probability and the winning probability of a random guess: $p_q - \frac{1}{2}$ for quantum strategies and $p_{cl} - \frac{1}{2}$ for classical strategies. The advantage of the quantum strategies is then measured by the ratio $\frac{p_q - 1/2}{p_{cl} - 1/2}$. Then, the maximum advantage that quantum strategies can achieve for 2-player XOR games is exactly equal to *Grothendieck's constant* K_G [9], a well known mathematical constant that satisfies $1.67696 \dots \leq K_G \leq 1.78221 \dots$ [11,15,4].

We also know that the maximum success probability p_q of quantum players in a 2-player XOR game can be described by a semidefinite program [8] and, hence, can be calculated in polynomial time [7]. In contrast, computing the classical success probability p_{cl} is NP-hard.

For XOR games with more than two players, several examples of games providing a quantum advantage are known. Mermin and Ardehali [13,3] have constructed an N -player XOR game with binary inputs $x_1, \dots, x_N \in \{0, 1\}$ with a quantum advantage of $2^{\lfloor \frac{N}{2} \rfloor - \frac{1}{2}}$. Perez-Garcia et al. [14] and Briët and Vidick [5] have shown that an arbitrarily large quantum advantage can be achieved with just 3 players — at the cost of increasing the number of possible values for the questions x_i . The best result is an advantage of $\Theta(\sqrt[m]{m})$ with m -valued questions x_1, x_2, x_3 [5].

There is also a substantial amount of theory of N -partite nonlocal games in the framework of Bell inequalities [16–18]. However, very little is known about algorithms for efficiently computing the best winning probability for an arbitrary N -party XOR game (or even an arbitrary N -party XOR game from a sufficiently broad class of games).

1.3. Our results

In this paper, our goal is to identify subclasses of XOR games for which it is easy to compute the winning probability in both classical and quantum case. We consider XOR games with N players, where each player receives a one-bit input $x_i \in \{0, 1\}$.

In the Section 2, we give methods for calculating the classical and quantum winning probabilities for the special case when the rules of the game are symmetric with respect to permuting the players. Our method for calculating the classical winning probability is based on elementary combinatorics. The method for calculating the quantum winning probability is obtained by simplifying the approach of [16,17] in the case of symmetric XOR games to obtain a particularly simple expression for the winning probability.

Then, in the Section 3 we apply our methods to the particular case of Ardehali's game [3] (derived from the Mermin–Ardehali inequalities [13,3]). Here, our results coincide with [3] but are obtained using different methods (which are more combinatorial in their nature). The advantage of our methods is that they can be easily applied to any symmetric XOR game, while those of [3] are tailored to that particular XOR game.

Finally, in the Section 4 we apply a similar technique for analyzing the generalized version of CHSH game, where the referee's choice is not restricted with the uniform input distribution. This overlaps with the work of Lawson et al. [12] which also analyzes CHSH game under non-uniform distributions on the inputs. We, however, analyze different probability distributions and use different methods.

Another application of our methods from Section 2.3 has been recently provided by [1] who used them to analyze several symmetric XOR games in a worst-case scenario.

2. Nonlocal XOR games

2.1. Introduction and definition

As described in the previous section, we consider nonlocal games of N players. Players receive a uniformly random input $x_1, \dots, x_N \in \{0, 1\}$ with the i^{th} player receiving x_i . The i^{th} player then produces an output $y_i \in \{0, 1\}$. No communication is allowed between the players, but they can use shared randomness (in the classical case) or quantum entanglement (in the quantum case). Players win if a winning condition $f(x_1, \dots, x_N, y_1, \dots, y_N)$ is satisfied.

We can describe the winning condition by a sequence of 2^N elements $[(I_{00\dots 0}, I_{00\dots 1}, \dots, I_{11\dots 1})]$, where each element corresponds to a sequence of inputs $(x_1, \dots, x_N)$ and describes the right answers for that input: $I_{x_1\dots x_N}$ is a set of all $(y_1, \dots, y_N) \in \{0, 1\}^N$ that satisfy $f(x_1, \dots, x_N, y_1, \dots, y_N)$.

For each $I_{x_1 x_2 \dots x_N}$, there are 2^{2^N} possible values. Therefore, there are $\binom{2^{2^N}}{2^N} = 2^{2^{2^N}}$ different games. This means 65536 games for $N = 2$, $\approx 1.8 \cdot 10^{19}$ games for $N = 3$ and practically innumerable for $N > 3$.

We will concentrate on those games, which are symmetric with respect to permuting the players and whose outcomes depend only on parity of the sum of the output (or Hamming weight of the output), i.e. on $XOR(|OUTPUT|)$. (Actually, this decision was based not on strict analytics, but rather on the results of numerical experiments: XOR games seem to be the most interesting in their quantum versions.)

Each such XOR game can be described as a string of $N + 1$ bits: $P_0 P_1 \dots P_N$, where each bit P_i represents the correct parity of the output sum in the case when the sum of input is i . We use “+” to denote the case when the correct parity is even and “−” to denote the case when the correct parity is odd.

For example, for the CHSH case, the correct answer is even if $|INPUT| = 0$ or 1 and odd if $|INPUT| = 2$. In our notation, we would describe it as “+ + −”.

2.2. Analysis of classical XOR games

Imagine a classical version of XOR game for which we want to find optimal classical strategies for players. Each player has four different choices – (00), (01), (10), (11). (The first bit here represents the answer on input 0, and second bit represents the answer on input 1. Thus, (ab) denotes a choice to answer a on input 0 and answer b on input 1.)

Definition 1 (Classical Normalized Strategy). Classical normalized strategy for an N -player XOR game is one of the following $2N + 2$ choice sequences:

$$\begin{aligned} & (00)^{N-k} (01)^k \\ & (00)^{N-1} (11) \\ & (00)^{N-k} (01)^{k-1} (10), \end{aligned}$$

where $k \in \{0, 1, \dots, N\}$.

Theorem 1. For any classical strategy for an N -player XOR game there exists a normalized strategy, such that these strategies on equal input answer equal parity.

Proof. First of all, remember, that we consider only symmetric games with respect to players permutation. Therefore, we always will order players by their choices.

The second step is choice inversion for a pair of players. If we take any pair of choices and invert both of them, the parity of the output will not change. Thus, we can find the following pairs of choices and make corresponding inversions:

$$\begin{aligned} (11) (11) & \rightarrow (00) (00) \\ (11) (10) & \rightarrow (00) (01) \\ (11) (01) & \rightarrow (00) (10) \\ (10) (10) & \rightarrow (01) (01) \end{aligned}$$

If it is impossible to find such pair, there is clearly no more than one choice from the set $\{(10), (11)\}$, and the presence of choice (11) follows that all other choices are (00). In other words, this strategy is normalized. $\square$

This trick allows very efficient searching for an optimal strategy for classical version of a XOR game. Strategy of the form

$$(00)^{N-k} (01)^k$$

has outcome probability

$$O((00)^{N-k} (01)^k) = \frac{\sum_{\substack{0 \leq i \leq N \\ 0 \leq j \leq i \\ (j \text{ is odd}) \iff (i_i = +)}} \binom{N-k}{i-j} \binom{k}{j}}{2^N}.$$

All other normal strategies have outcomes computable as

$$\begin{aligned} O((00)^{N-1} (11)) & = 1 - O((00)^N) \\ O((00)^{N-k} (01)^{k-1} (10)) & = 1 - O((00)^k (01)^{N-k}). \end{aligned}$$

2.3. Analysis of quantum XOR games

Consider a possibly non-symmetric XOR game. Let $x_1, x_2, \dots, x_N$ be the inputs to the players. Define $c_{x_1, x_2, \dots, x_N} = 1$ if, to win for these inputs, players must output $y_1, y_2, \dots, y_N$ with XOR being 1 and $c_{x_1, x_2, \dots, x_N} = -1$ if players must output $y_1, y_2, \dots, y_N$ with XOR being 0.

Werner and Wolf [16,17] have shown that, for any strategy in quantum version of an XOR game, its bias (the difference between the winning probability p_{win} and the losing probability p_{los}) is equal to

$$f(\lambda_1, \lambda_2, \dots, \lambda_N) = \left| \frac{1}{2^N} \sum_{x_1, x_2, \dots, x_N \in \{0,1\}} c_{x_1, x_2, \dots, x_N} \lambda_1^{x_1} \lambda_2^{x_2} \dots \lambda_N^{x_N} \right| \quad (1)$$

for some $\lambda_1, \lambda_2, \dots, \lambda_N \in \mathbb{C}$ satisfying $|\lambda_1| = |\lambda_2| = \dots = |\lambda_N| = 1$. Conversely, for any such $\lambda_1, \lambda_2, \dots, \lambda_N$, there is a winning strategy with the bias being $f(\lambda_1, \lambda_2, \dots, \lambda_N)$.

The expression (1) is quite hard to optimize, because it involves a maximum over N variables λ_i . We now show that, for symmetric XOR games, it can be replaced by a simpler expression that involves optimization over just one variable λ .

Lemma 1. For symmetric XOR games, the maximum of $f(\lambda_1, \lambda_2, \dots, \lambda_N)$ is achieved when $\lambda_1 = \lambda_2 = \dots = \lambda_N$.

Proof. We fix all but two of λ_i . To simplify the notation, we assume that $\lambda_3, \dots, \lambda_N$ are the variables that have been fixed. Then, (1) becomes

$$a + b\lambda_1 + c\lambda_2 + d\lambda_1\lambda_2$$

for some a, b, c, d . Because of symmetry, we have $b = c$. Thus, we have to maximize

$$a + b(\lambda_1 + \lambda_2) + d\lambda_1\lambda_2. \quad (2)$$

Let $\lambda_1 = e^{i\theta_1}$ and $\lambda_2 = e^{i\theta_2}$. Let $\theta_+ = \frac{\theta_1 + \theta_2}{2}$ and $\theta_- = \frac{\theta_1 - \theta_2}{2}$. Then, (2) becomes

$$a + be^{i\theta_+}(e^{i\theta_-} + e^{-i\theta_-}) + de^{2i\theta_+} = A + B \cos \theta_-$$

where $A = a + de^{2i\theta_+}$ and $B = 2be^{i\theta_+}$. If we fix θ_+ , we have to maximize the expression $A + Bx$, $x \in [-1, 1]$. For any complex A, B , $A + Bx$ is either maximized by $x = 1$ (if the angle between A and B as vectors in the complex plane is at most $\frac{\pi}{2}$) or by $x = -1$ (if the angle between A and B is more than $\frac{\pi}{2}$). If $x = 1$, we have $\lambda_1 = \lambda_2 = e^{i\theta_+}$. If $x = -1$, we have $\lambda_1 = \lambda_2 = e^{-i\theta_+}$.

Thus, if $\lambda_1 \neq \lambda_2$, then the value of (1) can be increased by keeping the same $\theta_+ = \frac{\theta_1 + \theta_2}{2}$ but changing λ_1 and λ_2 so that they become equal. The same argument applies if $\lambda_i \neq \lambda_j$. $\square$

Thus, we can find the value of a symmetric XOR game by maximizing

$$f(\lambda) = \left| \frac{1}{2^N} \sum_{k=0}^N \binom{N}{k} c_k \lambda^k \right| \quad (3)$$

where $c_k = 1$ if $P_k = "+"$ and $c_k = -1$ if $P_k = "-"$. The maximal $f(\lambda)$ is the maximum possible gap $p_{win} - p_{los}$ between the winning probability p_{win} and the losing probability p_{los} . We have $p_{win} = \frac{1 + f(\lambda)}{2}$ and $p_{los} = \frac{1 - f(\lambda)}{2}$.

Comparison with previous work. The formula (1) has been known, but its simplification (Lemma 1 and formula (3)) is new. In the next sections, we will see that this simplification makes it easy to calculate the maximum quantum winning probability in symmetric XOR games.

3. Mermin–Ardehali game

3.1. Introduction

There are four quantum XOR games (equivalent to each other up to the input and/or output inversion), which give the biggest gap between “classical” and “quantum” outcomes. Those games were discovered in the context of Bell inequalities (a physics notion closely related to nonlocal games) by Ardehali [3], building on an earlier work by Mermin [13]. They can be described as follows:

$ INPUT $	0	1	2	3	...	N	
$XOR(OUTPUT)$	+	+	−	−	...		$\begin{cases} + \text{ if } N \bmod 4 \in \{0, 1\} \\ - \text{ otherwise} \end{cases}$
$ INPUT $	0	1	2	3	...	N	
$XOR(OUTPUT)$	+	−	−	+	...		$\begin{cases} + \text{ if } N \bmod 4 \in \{0, 3\} \\ - \text{ otherwise} \end{cases}$
$ INPUT $	0	1	2	3	...	N	
$XOR(OUTPUT)$	−	−	+	+	...		$\begin{cases} + \text{ if } N \bmod 4 \in \{2, 3\} \\ - \text{ otherwise} \end{cases}$
$ INPUT $	0	1	2	3	...	N	
$XOR(OUTPUT)$	−	+	+	−	...		$\begin{cases} + \text{ if } N \bmod 4 \in \{1, 2\} \\ - \text{ otherwise} \end{cases}$

For each of those games, the maximum winning probability is $p_q = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ for a quantum strategy and $p_{cl} = \frac{1}{2} + \frac{1}{2^{\lceil \frac{N+1}{2} \rceil}}$ for a classical strategy [3]. Thus, if we take the ratio $\frac{p_q - 1/2}{p_{cl} - 1/2}$ as the measure of the quantum advantage,

these games achieve the ratio of $2^{\lfloor \frac{N}{2} \rfloor - \frac{1}{2}}$. Werner and Wolf [16] have shown that this is the best possible ratio. A similar ratio was earlier achieved by Mermin [13] for a partial XOR game:

$ INPUT $	0	1	2	3	...	N	
$XOR(OUTPUT)$	+	any	−	any	...		$\begin{cases} + \text{ if } N \bmod 4 \in \{0\} \\ - \text{ if } N \bmod 4 \in \{2\} \end{cases}$

In this game, the input of the players is chosen uniformly at random among all inputs with an even number of 1s. We now derive the winning probabilities for Ardehali's game using our methods.

3.2. Classical version

As all of these games are symmetric to each other, we will consider only the first game (4). Any normalized strategy for a classical version of such game can be further simplified.

Lemma 2. *In Ardehali's game, if there are two players with strategies (01) and (01), replacing their strategies by (00) and (11) does not change the average winning probability.*

Proof. Let us compare a strategy outcome before and after a simplification of type (01) (01) $\rightarrow$ (00) (11). Imagine a situation, in which the referee has already produced inputs for all except two players. The referee is ready to toss up his coin twice in order to decide, what input to give to the remaining players. Even if the coin produces different inputs for these players, their answers will be the same: one will answer 0 and other will answer 1, so the outcome will be unchanged.

If the coin produces equal inputs for players — 00 or 11 — the case requires more observations. First, the rules of the game require different answers for 00 and for 11. This can be seen from Table (4): changing $|INPUT|$ by 2, right answer changes to opposite value. Second, the strategy before the simplification resulted in equal answers on input 00 and on input 11 — one correct and one incorrect answer. Third, the strategy after the simplification will do the same (but in opposite sequence): one answer will be incorrect and other will be correct. So, the total average is equal for both strategies. $\square$

When none of the simplifications can be applied to the strategy, the strategy will be one of the following set:

$$\begin{aligned} & (00)^N \\ & (00)^{N-1} (01) \\ & (00)^{N-1} (10) \\ & (00)^{N-1} (11) \end{aligned}$$

Theorem 2. *For any N the optimal classical strategy has winning probability $\frac{1}{2} + 2^{-\lceil \frac{N+1}{2} \rceil}$.*

Proof. By $f(N, 01)$ we will denote the number of winning inputs (as a subset of all 2^N inputs) for N -player game with strategy $(00)^{N-1}(01)$. The value of f for $f(N, 00)$, $f(N, 10)$, $f(N, 11)$ will be defined similarly. We will use these two elementary facts extensively: $\binom{N}{k} = 0$ if k exceeds N or is less than zero, and $\binom{N}{k} + \binom{N}{k-1} = \binom{N+1}{k}$, which holds for any pair of N and k . Let us evaluate $f(N, 00)$. This strategy always produces XOR equal to zero, therefore $f(N, 00)$ is equal to the number of all N -bit strings, whose Hamming weight is 0 or 1 modulo 4. Counting over all Hamming weights we have $f(N, 00) = \binom{N}{0} + \binom{N}{4} + \binom{N}{8} + \dots = \sum_{k=0}^{\infty} \binom{N}{4k} = \sum_{k=0}^{\infty} \binom{N+1}{4k+1}$ (this infinity-based notation will be helpful later). The sum is bounded because of the first elementary fact.

Strategy $f(N, 01)$ always produces a XOR equal to the last bit of the input. If the last bit is 0, then for strategy to give a correct answer, the Hamming weight of all other bits of the input must be 0 or 1 modulo 4, and there are $\sum_{k=0}^{\infty} \binom{N-1}{4k} + \binom{N-1}{4k+1} = \sum_{k=0}^{\infty} \binom{N}{4k+1}$ such inputs. If the last bit is 1, then for the strategy to give a correct answer, the Hamming weight of all other bits of the input must be 1 or 2 modulo 4, and there are $\sum_{k=0}^{\infty} \binom{N-1}{4k+1} + \binom{N-1}{4k+2} = \sum_{k=0}^{\infty} \binom{N}{4k+2}$ such inputs. Taking these two facts together we have: $f(N, 01) = \sum_{k=0}^{\infty} \binom{N}{4k+1} + \sum_{k=0}^{\infty} \binom{N}{4k+2} = \sum_{k=0}^{\infty} \binom{N+1}{4k+2}$. Analyzing the other two cases in a similar manner we get $f(N, 10) = \sum_{k=0}^{\infty} \binom{N+1}{4k}$ and $f(N, 11) = \sum_{k=0}^{\infty} \binom{N+1}{4k+3}$. Let us notice that $f(1, 00) = 2, f(1, 01) = 1, f(1, 10) = 1, f(1, 11) = 0$. Also notice that $f(N+1, 00) = f(N, 00) + f(N, 10), f(N+1, 01) = f(N, 01) + f(N, 00), f(N+1, 10) = f(N, 10) + f(N, 11)$ and $f(N+1, 11) = f(N, 11) + f(N, 01)$, which can be proved by using the second elementary fact. By using mathematical induction one can immediately prove that the number of winning inputs for each of strategies can be characterized as follows:

$N \pmod 8$	(00)	(01)	(10)	(11)
0	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$
1	$2^{\frac{N-1}{2}} + 2^{N-1}$	2^{N-1}	2^{N-1}	$2^{N-1} - 2^{\frac{N-1}{2}}$
2	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$
3	2^{N-1}	$2^{\frac{N-1}{2}} + 2^{N-1}$	$2^{N-1} - 2^{\frac{N-1}{2}}$	2^{N-1}
4	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$
5	$2^{N-1} - 2^{\frac{N-1}{2}}$	2^{N-1}	2^{N-1}	$2^{\frac{N-1}{2}} + 2^{N-1}$
6	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{N-1} - 2^{\frac{N}{2}-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$	$2^{\frac{N}{2}-1} + 2^{N-1}$
7	2^{N-1}	$2^{N-1} - 2^{\frac{N-1}{2}}$	$2^{\frac{N-1}{2}} + 2^{N-1}$	2^{N-1}

This proves for any N the optimal strategy has $2^{\lfloor \frac{N-1}{2} \rfloor} + 2^{N-1}$ winning inputs and therefore the winning probability is $\frac{1}{2} + 2^{-\lfloor \frac{N+1}{2} \rfloor}$. $\square$

Comparison with previous work. Our analysis above is different from the one in Ardehali's paper [3]. Ardehali's analysis is based on a clever trick that reduces the winning probability to an expression involving complex numbers. This results in a proof that is fairly short but is specific to Ardehali's game. Our method of evaluating a sum of binomial coefficients is applicable to any symmetric XOR game.

3.3. Quantum version

The value of the Ardehali's game can be obtained by maximizing the one-variable expression in equation (3). In the case of Ardehali's game, the maximum of this expression is $\frac{1}{\sqrt{2}}$ and it is achieved by $\lambda = e^{i\theta}$ where $\theta = \frac{(2N+1) \bmod 8}{N} \pi + k \frac{2\pi}{N}$.

The result $f(\lambda) = \frac{1}{\sqrt{2}}$ corresponds to the winning probability of $p_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$. This has been known since [3] that $p_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ but the advantage of our approach is that we obtain the maximum winning probability by an easy argument that is applicable to any symmetric XOR game.

The winning strategy can also be obtained by reversing the argument of [16] and going from λ to transformations for the N players. There are infinitely many possible sets of strategies for each of the given θ . One of these strategies is described in [3]. We include an example of another strategy – of course, a symmetric one – in the next subsection (3.4).

The optimality of $p_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ can be also shown by a very simple argument, which does not involve any of the machinery above.

Theorem 3. $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ is the best possible winning probability for quantum strategy.

Proof. We modify the game by providing the inputs and the outputs of the first $N-2$ players to the $(N-1)^{\text{st}}$ and N^{th} players. Clearly, this makes the game easier: the last two players can still use the previous strategy, even if they have the extra knowledge.

Let k be the number of 1s among the first $N-2$ inputs. Then, we have the following dependence of the result on the actions of the last two players.

$x_1 + x_2 + \dots + x_{N-2}$	$x_{N-1} + x_N$		
	0	1	2
k	+	+	– if $k \bmod 4 = 0$
k	+	–	– if $k \bmod 4 = 1$
k	–	–	+
k	–	+	+

In each of the four cases, we get a game (for the last two players) which is equivalent to the CHSH game and, therefore, cannot be won with probability more than $\frac{1}{2} + \frac{1}{2\sqrt{2}}$. $\square$

The argument above is a particularly simple way of proving $p_{win} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}}$. To the best of our knowledge, this proof has not been found before.

3.4. Optimal quantum strategy

A common behavior for a player in a quantum nonlocal game is to perform some local operation on his qubit, perform a measurement in the standard basis and answer the result of the measurement. In other words, a choice for a player can be expressed with two matrices: one for input 0 and other for input 1. In fact, players may use equal strategies to achieve the best outcome. So an optimal strategy for any quantum XOR game can be found and proved with numerical optimization quite simply. For quantum version of Ardehali's game, the two matrices for all players look like the following:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \text{vector at } \gamma & 1 \\ -1 & \text{vector at } \gamma \end{pmatrix} \text{ for input 0; } \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \text{vector at } \gamma & 1 \\ -1 & \text{vector at } \gamma \end{pmatrix} \text{ for input 1}$$

Complex numbers here are illustrated as vectors on the complex plane. Marked small angles between vectors and axes are equal to each other and their value γ depends on the number of players.

Assuming x as value of input bit, these two matrices can be described as one:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{\pi}{2} \cdot (1-x) + \gamma)} & e^0 \\ e^{i\pi} & e^{i(-\frac{\pi}{2} \cdot (1-x) - \gamma)} \end{pmatrix} \quad (5)$$

The value γ can be expressed by the table:

Number of players	Angle γ
2	$\frac{5}{8}\pi$
3	$\frac{7}{12}\pi$
4	$\frac{1}{16}\pi$
5	$\frac{3}{20}\pi$
6	$\frac{5}{24}\pi$
7	$\frac{7}{28}\pi$
8	$\frac{1}{32}\pi$
$\vdots$	$\vdots$
N	$\frac{(2N+1) \bmod 8}{4N}\pi$
$\vdots$	$\vdots$

Consider a quantum system in starting state $|GHZ\rangle = \frac{1}{\sqrt{2}}|00\dots 0\rangle + \frac{1}{\sqrt{2}}|11\dots 1\rangle$. Let us express all local operations that players apply to their qubits during the game as a tensor product of N matrices $M = C_1 \otimes C_2 \otimes \dots \otimes C_N$ (where each C_k can be represented as in (5)). Then each cell of M can be calculated as follows:

$$M_{[j_1 j_2 \dots j_N, i_1 i_2 \dots i_N]} = \prod_{k=1}^N C_{k[j_k, i_k]}$$

where $i_1, i_2, \dots, i_N, j_1, j_2, \dots, j_N \in \{0, 1\}$.

After all local operations are complete, let us express the final state directly as sum of its amplitudes

$$\sum_{y_1, \dots, y_N \in \{0, 1\}} \alpha_{y_1 \dots y_N} |y_1 \dots y_N\rangle = M \left(\frac{1}{\sqrt{2}}|00\dots 0\rangle + \frac{1}{\sqrt{2}}|11\dots 1\rangle \right)$$

Consider a value of arbitrary amplitude $\alpha_{y_1 \dots y_N}$. As there are only two nonzero amplitudes in the starting state, any $\alpha_{y_1 \dots y_N}$ will consist of two summands:

$$\alpha_{y_1 \dots y_N} = \frac{1}{\sqrt{2}} \prod_{k=1}^N C_{k[0, y_k]} + \frac{1}{\sqrt{2}} \prod_{k=1}^N C_{k[1, y_k]}$$

Table 1
Amplitude angle values for different inputs.

$ INPUT = x_1 + x_2 + \dots + x_N$	$ OUTPUT = y_1 + y_2 + \dots + y_N$				
	0	1	2	3	...
0	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	...
1	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	...
2	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	...
3	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	...
4	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Assuming the players got N -bit input $x_1 \dots x_N$, let us substitute values from (5) for each C_k :

$$\begin{aligned} \alpha_{y_1 \dots y_N} &= \frac{1}{\sqrt{2}} \prod_{k=1}^N \frac{1}{\sqrt{2}} e^{i((\gamma + \frac{\pi}{2}(1-x_k)) \cdot (1-y_k))} + \frac{1}{\sqrt{2}} \prod_{k=1}^N \frac{1}{\sqrt{2}} e^{i(\pi + (\frac{\pi}{2} - \gamma + \frac{\pi}{2} \cdot x_k) \cdot y_k)} \\ &= \left(\frac{1}{\sqrt{2}}\right)^{N+1} e^{i \sum_{k=1}^N (\gamma + \frac{\pi}{2}(1-x_k)) \cdot (1-y_k)} + \left(\frac{1}{\sqrt{2}}\right)^{N+1} e^{i \sum_{k=1}^N (\pi + (\frac{\pi}{2} - \gamma + \frac{\pi}{2} \cdot x_k) \cdot y_k)} \end{aligned}$$

Now we are mainly interested in the difference between rotation angles on the complex plane for these two summands. That is,

$$\begin{aligned} &\sum_{k=1}^N \left[\left(\gamma + \frac{\pi}{2}(1-x_k) \right) (1-y_k) - \left(\pi + \left(\frac{\pi}{2} - \gamma + \frac{\pi}{2} x_k \right) y_k \right) \right] \\ &= \sum_{k=1}^N \left[\left(\gamma + \frac{\pi}{2} - \frac{\pi}{2} x_k - \gamma y_k - \frac{\pi}{2} y_k + \frac{\pi}{2} x_k y_k \right) - \left(\pi + \frac{\pi}{2} y_k - \gamma y_k + \frac{\pi}{2} x_k y_k \right) \right] \\ &= \sum_{k=1}^N \left(\gamma + \frac{\pi}{2} - \frac{\pi}{2} x_k - \frac{\pi}{2} y_k - \pi - \frac{\pi}{2} y_k \right) \\ &= \sum_{k=1}^N \left(\gamma - \frac{\pi}{2} - \frac{\pi}{2} x_k - \pi y_k \right) \end{aligned}$$

Let us now consider the case of $N = 4n$ and $\gamma = \frac{1}{4N}\pi$ (and similar reasoning holds for any N). In this case the difference is expressed by

$$\frac{1}{4}\pi - \frac{1}{2}\pi \sum_{k=1}^N (x_k + 2y_k),$$

which is obtained by throwing out $\frac{\pi}{2} \times N \equiv 0 \pmod{2\pi}$, which is now redundant. However, by modulus 2π this expression is equal to value from Table 1.

If the angle between two summands (both of the same length $\left(\frac{1}{\sqrt{2}}\right)^{N+1}$) is $\boxed{\pm \frac{1}{4}\pi}$, then their sum is

$$\left(\frac{1}{\sqrt{2}}\right)^{N-1} \cos \frac{\pi}{8} = \left(\frac{1}{\sqrt{2}}\right)^{N-1} \frac{\sqrt{2+\sqrt{2}}}{2} \quad (6)$$

(Recall that the absolute value of the sum of two vectors of the same length x with angle ϕ between them is calculated as $|2x \cos \frac{\phi}{2}|$.)

If the angle between two summands (both of the same length $\left(\frac{1}{\sqrt{2}}\right)^{N+1}$) is $\pm \frac{3}{4}\pi$, then their sum is

$$\left(\frac{1}{\sqrt{2}}\right)^{N-1} \cos \frac{3\pi}{8} = \left(\frac{1}{\sqrt{2}}\right)^{N-1} \frac{\sqrt{2-\sqrt{2}}}{2}$$

As one can see from Table 1, bigger amplitudes always correspond to correct answers (and smaller amplitudes correspond to incorrect answers). The sum of the squares of formula (6), i.e. the measurement result for any fixed input, will give the probability of the right answer:

$$\sum_{\text{Angle}=\pm\frac{1}{4}\pi} \left(\left(\frac{1}{\sqrt{2}} \right)^{N-1} \frac{\sqrt{2+\sqrt{2}}}{2} \right)^2 = 2^{N-1} \left(\frac{1}{\sqrt{2}} \right)^{2(N-1)} \left(\frac{\sqrt{2+\sqrt{2}}}{2} \right)^2$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Note that this probability is stable: it remains the same for all possible inputs from the referee.

4. CHSH game with arbitrary input distribution

4.1. Introduction

The CHSH game is a popular illustration of difference between classical and quantum worlds. Namely, it shows a violation of the original Bell's inequality. In terms of the CHSH game, the biggest such violation can be achieved by choosing uniformly distributed players' input.

However, it could be interesting to study a more general case, i.e. CHSH games with different input distributions.

4.2. Classical version

Let us start with describing of optimal strategy for a classical CHSH game with arbitrary distribution of input.

First of all it is worth recalling the fact that no randomized strategy can give better results than optimal deterministic one does. Therefore, it is sufficient to consider and evaluate a finite number of different deterministic strategies only. Since each of the two players has to determine his binary answer for two possible inputs, this number is $2^4 = 16$. However, it turns out to be sufficient to consider only four of them in order to find the optimum.

The following four strategies belong to the set of optimal classical strategies for CHSH game with uniform distribution of input.

Answers of player 1		Answers of player 2		Inputs, for which strategy gives correct answer
Input 0	Input 1	Input 0	Input 1	
0	0	0	0	0, 0 0, 1 1, 0 ~~1, 1~~
0	1	0	0	0, 0 0, 1 ~~1, 0~~ 1, 1
0	0	0	1	0, 0 ~~0, 1~~ 1, 0 1, 1
0	1	1	0	~~0, 0~~ 0, 1 1, 0 1, 1

Since each strategy in the CHSH game results in an incorrect answer for at least one input, each of other 12 strategies either is equivalent to one of the above-mentioned four strategies or is strictly dominated by one of them. Therefore it is sufficient to consider only the four above-mentioned strategies.

Suppose input distribution $(P_{00}, P_{01}, P_{10}, P_{11})$, where P_{ab} stands for probability to get input “a, b” from the referee. We must select any pair (a^*, b^*) , such that

$$P_{a^*b^*} = \min_{a,b} P_{ab}$$

Then we must search for this pair crossed out in the rightmost column of the table (i.e. for “~~1, 1~~”) and select the respective strategy. This strategy will exploit differences of input distributions in the best possible way and will give outcome

$$1 - \min(P_{00}, P_{01}, P_{10}, P_{11})$$

which is evidently the best possible expected value for given distribution.

We conclude that classical outcome depends linearly on input distribution and has the lowest expected value (0.75) for uniform input distribution.

4.3. Quantum version

In this section we only consider those input distributions which preserve symmetry of the game with respect to permutations of players. In other words, we shall assume that $P_{01} = P_{10} = \frac{1 - P_{00} - P_{11}}{2}$. Our result is

Theorem 4. The maximum achievable winning probability is equal to

$$p_{win} = \begin{cases} 1 - P_{00} & \text{if } \frac{(1-P_{00})}{4P_{11}} - \frac{(1-P_{11})}{4P_{00}} \leq -1 \\ 1 - P_{11} & \text{if } \frac{(1-P_{00})}{4P_{11}} - \frac{(1-P_{11})}{4P_{00}} \geq 1 \\ \frac{1}{2} + \frac{P_{00} + P_{11}}{4} \sqrt{\frac{(P_{00} + P_{11})^2 + (1 - 2P_{00})(1 - 2P_{11})}{P_{00}P_{11}}} & \text{otherwise} \end{cases}$$

Proof. In order to find optimal quantum strategy for players in such a symmetric game, we can make some useful assumptions:

Lemma 3. Assume that the probability distribution on the inputs is symmetric (i.e., satisfies $P_{01} = P_{10}$). Then, there is a strategy for the players that achieves the maximum success probability and satisfies the following requirements:

1. Players share quantum system in state $|GHZ\rangle = \frac{1}{\sqrt{2}}|00\dots 0\rangle + \frac{1}{\sqrt{2}}|11\dots 1\rangle$.
2. Given x as an input, a player applies operation

$$\hat{U}_x = \begin{pmatrix} \frac{e^{i\phi_x}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & e^{-i\phi_x} \end{pmatrix}$$

to his qubit and then make a measurement in the computational basis. Players always answer with this result of the measurement.

3. The strategies for both players are the same.

Proof. 1. The first fact is proven in [16, section V.D].

2. The second fact also simply follows from [16]. The transformations $\hat{U}_x$ and a measurement in the computational basis correspond to observables

$$A(x) = \hat{U}_x^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \hat{U}_x = \begin{pmatrix} 0 & -e^{i\phi_x} \\ -e^{-i\phi_x} & 0 \end{pmatrix}$$

[16, section V.D] shows that these observables give the optimal strategy.

3. Equality of players' strategies could be proven in the similar way as in analysis of general XOR games. $\square$

There are two possible ways to figure out the best game outcome for the described set of quantum strategies.

The first is as follows. Given an input “ x, y ”, players perform the following transformation on their entangled quantum system:

$$\begin{aligned} \hat{U}_x \otimes \hat{U}_y &= \begin{pmatrix} \frac{e^{i\phi_x}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & e^{-i\phi_x} \end{pmatrix} \otimes \begin{pmatrix} \frac{e^{i\phi_y}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & e^{-i\phi_y} \end{pmatrix} \\ &= \begin{pmatrix} \frac{e^{i(\phi_x+\phi_y)}}{2} & \frac{e^{i\phi_x}}{2} & \frac{e^{i\phi_y}}{2} & \frac{1}{2} \\ -\frac{e^{i\phi_x}}{2} & \frac{e^{i(\phi_x-\phi_y)}}{2} & -\frac{1}{2} & e^{-i\phi_y} \\ -\frac{e^{i\phi_y}}{2} & \frac{1}{2} & \frac{e^{i(\phi_y-\phi_x)}}{2} & e^{-i\phi_x} \\ \frac{1}{2} & -\frac{e^{-i\phi_y}}{2} & -\frac{e^{-i\phi_x}}{2} & \frac{e^{-i(\phi_x+\phi_y)}}{2} \end{pmatrix} \end{aligned}$$

We only have to consider amplitudes of $|00\rangle$ and $|11\rangle$ of initial state and on the same amplitudes of final state, that is, the corner values of this matrix.

The winning probability for such strategy is as follows:

$$\begin{aligned} \text{Win}(\phi_0, \phi_1) = & P_{00} \times \left(\left| \frac{e^{i(\phi_0+\phi_0)}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2\sqrt{2}} + \frac{e^{-i(\phi_0+\phi_0)}}{2\sqrt{2}} \right|^2 \right) + \\ & P_{01} \times \left(\left| \frac{e^{i(\phi_0+\phi_1)}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2\sqrt{2}} + \frac{e^{-i(\phi_0+\phi_1)}}{2\sqrt{2}} \right|^2 \right) + \\ & P_{10} \times \left(\left| \frac{e^{i(\phi_1+\phi_0)}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right|^2 + \left| \frac{1}{2\sqrt{2}} + \frac{e^{-i(\phi_1+\phi_0)}}{2\sqrt{2}} \right|^2 \right) + \\ & P_{11} \times \left(1 - \left| \frac{e^{i(\phi_1+\phi_1)}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \right|^2 - \left| \frac{1}{2\sqrt{2}} + \frac{e^{-i(\phi_1+\phi_1)}}{2\sqrt{2}} \right|^2 \right) \end{aligned}$$

Since $|e^{i\phi} + 1| = |e^{-i\phi} + 1| = 2 \cos \frac{\phi}{2}$, we can rewrite this expression as

$$\begin{aligned} \text{Win}(\phi_0, \phi_1) = & P_{00} \cos^2 \phi_0 + \\ & (P_{01} + P_{10}) \cos^2 \frac{\phi_0 + \phi_1}{2} + \\ & P_{11} - P_{11} \cos^2 \phi_1 \end{aligned}$$

Since $\cos^2 \phi = \frac{\cos 2\phi + 1}{2}$,

$$\begin{aligned} \text{Win}(\phi_0, \phi_1) = & \frac{P_{00}}{2} (\cos 2\phi_0 + 1) + \\ & \frac{P_{01} + P_{10}}{2} (\cos (\phi_0 + \phi_1) + 1) + \\ & P_{11} - \frac{P_{11}}{2} (\cos 2\phi_1 + 1) \\ = & \frac{P_{00}}{2} \cos 2\phi_0 + \frac{P_{01} + P_{10}}{2} \cos (\phi_0 + \phi_1) - \frac{P_{11}}{2} \cos 2\phi_1 + \\ & \underbrace{\frac{P_{00} + P_{01} + P_{10} + P_{11}}{2}}_{=\frac{1}{2}} \end{aligned}$$

Finally, we define *Bias* (ϕ_0, ϕ_1) as *Win* (ϕ_0, ϕ_1) – *Loss* (ϕ_0, ϕ_1):

$$\begin{aligned} \text{Bias}(\phi_0, \phi_1) = & \text{Win}(\phi_0, \phi_1) - \text{Lose}(\phi_0, \phi_1) \\ = & \text{Win}(\phi_0, \phi_1) - (1 - \text{Win}(\phi_0, \phi_1)) \\ = & 2 \times \text{Win}(\phi_0, \phi_1) - 1 \\ = & P_{00} \cos 2\phi_0 + (P_{01} + P_{10}) \cos (\phi_0 + \phi_1) - P_{11} \cos 2\phi_1 \end{aligned}$$

We notice that this expression does not depend on ratio of P_{01} and P_{10} , but depends only on their sum. This is not surprising, since players use the same strategy, the expected value of the game must not depend on permutations of players. In further considerations it will be useful to replace $P_{01} + P_{10}$ with $1 - P_{00} - P_{11}$.

Now our task is: given values P_{00} and P_{11} , maximize the expression

$$\text{Bias}(\phi_0, \phi_1) = P_{00} \cos 2\phi_0 + (1 - P_{00} - P_{11}) \cos (\phi_0 + \phi_1) - P_{11} \cos 2\phi_1 \quad (7)$$

This procedure turns to not be trivial. Extremum of function *Bias* (ϕ_0, ϕ_1) requires the following two conditions to be satisfied:

$$\begin{cases} \frac{d}{d\phi_0} \text{Bias}(\phi_0, \phi_1) = 0 \\ \frac{d}{d\phi_1} \text{Bias}(\phi_0, \phi_1) = 0 \end{cases}$$

It follows that

$$\begin{cases} -(1 - P_{00} - P_{11}) \sin (\phi_0 + \phi_1) - 2P_{00} \sin 2\phi_0 = 0 \\ -(1 - P_{00} - P_{11}) \sin (\phi_0 + \phi_1) + 2P_{11} \sin 2\phi_1 = 0 \end{cases}$$

And this could be rewritten in a slightly more beautiful form:

$$(1 - P_{00} - P_{11}) \sin (\phi_0 + \phi_1) = -2P_{00} \sin 2\phi_0 = 2P_{11} \sin 2\phi_1 \quad (8)$$

We then have to solve these equations, which is possible but complicated.

The second approach is to optimize the expression

$$|P_{00} + (1 - P_{00} - P_{11})\lambda - P_{11}\lambda^2| \quad (9)$$

which comes from (3).

Direct calculations show that

$$\max_{|\lambda|=1} |P_{00} + (1 - P_{00} - P_{11})\lambda - P_{11}\lambda^2| = \max_{|\lambda|=1} \sqrt{\begin{matrix} -4P_{00}P_{11}\Re(\lambda)^2 \\ +2(1 - P_{00} - P_{11})(P_{00} - P_{11})\Re(\lambda) \\ +1 - 2(1 - P_{00} - P_{11})(P_{00} + P_{11}) \end{matrix}}$$

(by $\Re(\lambda)$ we denote the real part of complex number λ).

The quadratic function of $\Re(\lambda)$ under the square root achieves its maximum at the point where its derivative $-8P_{00}P_{11}\Re(\lambda) + 2(1 - P_{00} - P_{11})(P_{00} - P_{11})$ turns to zero, that is, at

$$\Re(\lambda) = \frac{(1 - P_{00} - P_{11})(P_{00} - P_{11})}{4P_{00}P_{11}} = \frac{1 - P_{00}}{4P_{11}} - \frac{1 - P_{11}}{4P_{00}}$$

and the closer is $\Re(\lambda)$ to this point, the bigger is the value of the function.

Since $|\lambda| = 1 \implies \Re(\lambda) \in [-1, 1]$, the closest possible value for $\Re(\lambda)$ is at

$$\Re(\lambda_{\text{opt}}) = \max\left(-1, \min\left(1, \frac{(1 - P_{00} - P_{11})(P_{00} - P_{11})}{4P_{00}P_{11}}\right)\right) \quad (10)$$

The value of the game can be obtained by substituting λ_{opt} from (10) into (9):

$$\max_{\phi_0, \phi_1} \text{Bias} = \begin{cases} 1 - 2P_{00} & \text{if } \frac{(1-P_{00})}{4P_{11}} - \frac{(1-P_{11})}{4P_{00}} \leq -1 \\ 1 - 2P_{11} & \text{if } \frac{(1-P_{00})}{4P_{11}} - \frac{(1-P_{11})}{4P_{00}} \geq 1 \\ \frac{P_{00} + P_{11}}{2} \sqrt{\frac{(P_{00} + P_{11})^2 + (1 - 2P_{00})(1 - 2P_{11})}{P_{00}P_{11}}} & \text{otherwise} \end{cases}$$

Theorem 4 now follows from $\max_{\phi_0, \phi_1} \text{Bias} = 2p_{\text{win}} - 1$. $\square$

For most P_{00}, P_{11} the value of the quantum version of the game exceeds the value of the classical one, but at some intervals where $P_{00} \approx P_{11}$ these values coincide.

If we would like to find explicit observables achieving this bound, we can proceed as follows. According to [16], λ should be equal to the eigenvalue of $C = A(0)A(1)$. We have

$$C = A(0)A(1) = \begin{pmatrix} e^{i(\phi_0 - \phi_1)} & 0 \\ 0 & e^{i(\phi_1 - \phi_0)} \end{pmatrix} \quad (11)$$

In this way we can find the exact value of λ_{opt} , which corresponds to optimal eigenvalue of (11). Phase of this eigenvalue $\arg(\lambda_{\text{opt}})$ is just the difference between angles ϕ_0 and ϕ_1 :

$$\phi_1 = \phi_0 + \arg(\lambda_{\text{opt}}) \quad (12)$$

Therefore, we get

$$(1 - P_{00} - P_{11}) \sin(\phi_0 + \phi_0 + \arg(\lambda_{\text{opt}})) = -2P_{00} \sin 2\phi_0$$

Solving this equation will lead to the following result:

$$\sin 2\phi_0 = \pm \xi, \quad \text{where } \xi = \frac{1}{\sqrt{\left(\frac{\cos(\arg(\lambda_{\text{opt}})) + \frac{2P_{00}}{1-P_{00}-P_{11}}}{\sin(\arg(\lambda_{\text{opt}}))}\right)^2 + 1}} \quad (13)$$

Checking results for four possible values of ϕ_0 ($\frac{\pm \arcsin \xi}{2}$ and $\frac{\pi \mp \arcsin \xi}{2}$) will provide us with optimal ϕ_0 and ϕ_1 .

5. Conclusion

In this paper, we developed methods for computing the classical and the quantum winning probability in symmetric XOR games of N players (with binary inputs x_i) and applied them to two games: the Ardehali game (where we obtained the known results about winning probabilities in a new way and generalized similar results given in [2]) and the CHSH game with a non-uniform distribution on the inputs). A third application of our methods has been developed by [1] who used them to analyze several symmetric XOR games for the worst-case distribution on the inputs.

The big open question is: can we identify other classes of nonlocal games for which we can efficiently calculate the winning probability? For 2-party XOR games, the best quantum winning probability is computable in polynomial time via semidefinite programming [7] but very little is known about XOR games with more than 2 players, since Tsirelson's theorem [8] which is a basis for the semidefinite programming algorithm only applies to 2-player games.

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